SENIOR MATHEMATICAL OLYMPIAD 2025-2026

Time: 3 Hours Full Marks: 100

The symbols used have their usual meaning.
Answer all the questions.
All questions are of equal value.

1. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ numerically using the midpoint rule. What is the range of error?
2. What is the first digit of the expansion $(2+\sqrt{2})^{1000}$ after the decimal point ?
3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.7+sin n}}$ converges or diverges.
4. Determine the set of all positive real numbers a for which the inequality $a^x \geq x^a$ is true for all positive real numbers x.
5. Find the maximum value of xyz when $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
6. Suppose that $a_i \geq 0$ and $p_i \geq 0$ and $\sum_i p_i = 1$. Prove that $a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \leq a_1 p_1 + \cdots + a_n p_n$.
7. Let $f : [0,1] \to \mathbb{R}$ be continuous and $|f(x)| \leq \int_0^x f(t)dt \leq 0 \leq x \leq 1$. Show that $f \equiv 0$.
8. Can the interval $[0,1] \subset \mathbb{R}$ be written as union of countably infinite number of disjoint closed sets? Justify your answer.
9. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that at least two of them are in the wrong envelopes.
10. If $f(x+y) = f(x)f(y)$ for all real x, y , then prove that the function $\frac{f(x)}{1+|f(x)|^2}$ is an even function.
11. Find the value of the integral $\int_0^{\infty} \int_x^{\infty} \frac{1}{y} e^{\frac{-y}{2}} dydx$.
12. Let G be a group of order 231. Find the number of elements of order 11 in G.
13. If in a finite group G whose order is not divisible by 3 and $(ab)^3 = a^3 b^3 \;\; \forall \; a, \; b \in G$, then prove that the group G must be an abelian.
14. Find a group which contains a unique normal subgroup of order 4.
15. Let M be the space of all 4x3 matrices with entries in the finite field of 3 elements.Then find the number of matrices of rank 3 in M.
16. Let $C = \{z \in \mathbb{C} : |z| = 2\}$. Find the value of $\int_C \frac{z}{(9-z^2)(z+i)} dz = \frac{\pi}{5}$.
17. If U and V are null spaces of $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$ respectively, then find the dimension of $U+V$.
18. A sphere is inscribed in the tetrahedron with faces $x=0, y=0, z=0, 2x+6y+3z=14$. Find the equation of the sphere.
19. A fair die is thrown 600 times. Find a lower bound for the probability of getting 80 to 120 sixes.
20. Find the eigenvalues and eigen functions of $y'' + \lambda y = 0$ such that $y(0)=0, y(1)=0$.