Senior Mathematical Olympiad - 2022

December 11, 2022

(Answer all questions. Each of the questions carry 5 marks.)

Time: 3 hours | Full Marks: 100

(Answer all questions. Each question carries 5 marks)

1. Find two complex numbers C_1 and C_2 such that $1-i, 2i$ and $1+i$ are on the circle $\{Z: |Z+C_2| = C_1\}$.

2. (i) If in the group $G, x^5 = e, xyx^{-1} = y^2$ for $x, y \in G$, find $O(y)$.
   (ii) Find the smallest noncyclic abelian group of finite order.

3. What is the radius of convergence of the power series $\sum_{k=1}^{\infty} \sqrt{2\log_k x^k}$?

4. How many fields are there (upto isomorphism) with exacltly 6 elements ?

5. Obtain the integration formula $\int_0^1 \sqrt{x} f(x) dx \approx w_1 f(x_1) + w_2 f(x_2)$ considering $\sqrt{x}$ as a weight function such that the integration formula is exact for all polynomials of degree ≤ 3

6. Suppose f is defined in a neighbourhood of x and suppose $f''(x)$ exists. Show that; $\lim_{h-0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$

7. Prove that the sequence (x_n) defined by $x_1 = \sqrt{2}, x_{n+1} = \sqrt{(2+x_n)}$ is convergent. Hence find the limit.

8. Let A^* denote the transpose of the complex conjugate of a matrix A. If A is upper triangular matrix with complex entries and $A A^* = A^* A$, then show that A is a diagonal matrix.

9. If $\frac{sinx}{x}$ is a solution of the ordinary differential equation $xy'' + 2y' + xy = 0$ then find its other solution.

10. Probability of a defective watch manufactured by factory A is $\frac{1}{100}$ and factory B is $\frac{1}{200}$. A watch is drawn from a lot of A or B and is found to be good. What is the probability that the second watch drawn is also good.

11. How many points of discontinuity does the function,

$$f(x) = \begin{Bmatrix} \frac{1}{2^n}, \frac{1}{2^{(n+1)}} < X \le \frac{1}{2^n}, n = 0,1,2,\ldots\ldots \\ 0, otherwise. \end{Bmatrix}$$

have ? Is it integrable on [0,1] ? Justify your answer.

12. In the vector space $\mathbb{R}^3$, let $u=(1,2,1), v=(3,1,5), w=(3,-4,7)$ then what do you think about the subspaces spanned by S = $\{u,v\}$ and T = $\{u,v,w\}$? Are they same or different ? Justify your answer.

13. Let $p(z)=1+z+2z^2+3z^3+\dots\dots\dots+100z^{100}$ and $w=e^{\frac{\pi i}{100}}$ then find $p(1)+P(w)+p(w^2)+\dots\dots+p(w^{199})$.

14. Find the infinite series $\sum_{n=0}^{\infty} a_n$ such that $\lim_{n\to\infty} a_n=0$ but $\sum a_n$ divergent.

15. If $a_0+\frac{a_1}{2}+\frac{a_2}{3}+\dots\dots\dots\dots+\frac{a_n}{n+1}=0$ then, show that the polynomial $a_0+a_1x+a_2x^2+\dots\dots\dots\dots+a_nx^n$ has a root on (0,1).

16. In S_3 find two elements a , b such that $(ab)^3 \neq a^3 b^3$.

17. Let $T:R^3 \to R^2$ be a linear transform by $T(x,y,x)=(2y+z,-x+4y+5z)$ with the braces $\beta=$ {(1,1,0), (0,1,1), (1,0,1)}, γ = { (1,0), (0,1) }. Find the matrix representation of T.

18. Show that for all real numbers a and b; $\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$.

19. Find the examples of
(i) a sequence of rationals converging to irrational.
(ii) a sequence of irrational converging to rational.

20. Find the function $f:N \to N$ (Where N is natural number) which satisfies the relation $f(m+f(n))=n+f(m+2005)$ for all $m,n \in N$.

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