Senior Mathematical Olympiad - 2011

December 04, 2011

Time : 3 hours Full Marks : 100

Answer all questions

1. What whould be the reminder if $2009^{2010} + 2009^{2008}$ is divided by 2011 ? [8]

2. We have a spherical ball of radius r. A cylindrical hole is drilled through the ball with the axis of the cylinder along one of the diagonals of the sphere. If the height of the cylinder that is removed to get the hole is h, what volume of the ball would be left ? [6]

3. Show that
$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{ne}{\sqrt{2\pi}}^{-\frac{x^2 n^2}{2}} \cos x \, dx = 1$$ [8]

4. What geometrical object is $\{(x, y, z) : xy + yz + zx = 1\}$ [6]

5. Let P be a point on an ellipse with focii F_1 and F_2. Show that the normal to the ellipse at P bisects the angle $\angle F_1 P F_2$. [6]

6. Solve the differential equation
$$\frac{d^2x}{dt^2} + x = \cos t.$$ [6]

7. Give an example of a subset of $\mathbb{R}$ which has exactly two limit points. [6]

8. G is a group whose order is not divisible by 3 and $(ab)^3 = a^3b^3$ for a, $b \in G$. Show that G is abelian. [6]

9. Sum the series
$$1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots$$ [6]

10. If $a, b > 0$ $a + b = 1$

    Show that
$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}$$ [6]

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11. Can there be an analytic function with constant absolute value ? Justify your answer. [6]

12. Suppose 20 sweets are to be distributes among 9 children. In how many ways can this be done if every child is to get at least one. [6]

13. Establish necessary and sufficient condition on the constant k for the existence of *a* continuous real valued function f satisfying $f(f(x)) = k\,x^9$ for all real x. [8]

14. Let G be a group of 2 x 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a, b, c, d are integers modulo 3 such that $ad-bc \neq 0$, relative to matrix multiplication. Find $O(G)$. [6]

15. Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}\,|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, x_2, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, x_2, \ldots, x_{10}$.
(Here|S| means the number of elements in the set S.) [10]

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